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The effect of stable thermal stratification on the stability of viscous parallel flows

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A unified linear viscous stability theory is developed for a certain class of stratified parallel channel and boundary-layer flows with Prandtl number equal to unity. Results are presented for plane Poiseuille flow and the asymptotic suction boundary-layer profile, which show that the asymptotic behaviour of both branches of the curve of neutral stability has a universal character. For velocity profiles without inflexion points it is found that a mode of instability disappears as η, the local Richardson number evaluated at the critical point, approaches 0·0554 from below. Calculations for Grohne's inflexion-point profile show both major and minor curves of neutral stability for $0 < \eta \leqslant 0{\cdot}0554$; for

$$0{\cdot}0554 < \eta < 0{\cdot}0773$$

there is only a single curve of neutral stability; and, for $\eta > 0{\cdot}0773$, the curves of neutral stability become closed, with complete stabilization being achieved for a value of η of about 0·107.

1. Introduction

Considerable effort has been expended in recent years on the study of the stability of stably stratified inviscid parallel flows. Central to this work, reviewed by Drazin & Howard (1967), are the theorems of Rayleigh and Miles. Rayleigh's theorem,† which was derived for a homogeneous fluid, states that a *necessary* condition for instability is that the velocity profile possesses a point of inflexion. Miles's (1961) theorem states that a *sufficient* condition for stability of a stratified flow is that the local Richardson number exceed $\frac{1}{4}$ throughout the flow.

The conditionally destabilizing effect of viscosity through the action of the Reynolds stresses motivates a treatment of the viscous stability problem. A natural point of departure for this work is the stability theory involving the asymptotic solution ($R \rightarrow \infty$) of the Orr–Sommerfeld equation already developed for unstratified viscous parallel flows. Results which have been obtained from the asymptotic theory, reviewed by Reid (1965), are in good agreement with those obtained by direct numerical integration of the governing equations. At large Reynolds numbers, the asymptotic approach has the advantage of avoiding the difficulties intrinsic to the numerical methods.

† Rayleigh's theorem has been generalized to stratified fluids by Synge (1933) but his result is not of much use here.

Attempts to treat the viscous problem have been made by Schlichting and Koppel. Schlichting (1935), for example, considered the stability of a stratified boundary-layer profile. Although he retained the effect of viscosity, he neglected the effect of heat conduction, thereby restricting his analysis to a fluid of infinite Prandtl number. More recently, Koppel (1964) obtained integral representations for the solutions of the more general sixth-order equation with arbitrary Prandtl number. For values of the Prandtl number different from unity, the kernels in Koppel's integral representations involve Whittaker functions, and the analysis was not pursued further. By using the method of steepest descents, Koppel obtained asymptotic approximations to the solutions for a Prandtl number of unity but did not attempt to solve the characteristic value problem.

For a Prandtl number of 1, a significant simplification of the mathematical analysis for the viscous solutions of the governing equations occurs, since it is then possible to factor the sixth-order equation, and it is then sufficient to solve two third-order equations. This factorization was first recognized by Reid in connexion with the stability of spiral flow between rotating cylinders (Hughes & Reid 1968), and was later used by Gage & Reid (1968) in the investigation of the stability of thermally stratified plane Poiseuille flow. This paper presents a unified approach to the stability theory for stably stratified viscous parallel flows applicable to a large class of symmetric-channel and monotonic boundary-layer profiles.

2. Formulation of the problem

Consider a steady, thermally stratified, parallel flow of viscous, heat-conducting, incompressible fluid either in a boundary layer over a flat plate or in a channel. The direction of the flow will be denoted by the x co-ordinate and the vertical direction will be represented by the y co-ordinate. In order to avoid analytical difficulties associated with multiple critical layers, the velocity profile in the boundary layer or in the half-channel will be assumed monotonic.

All variables which appear in the governing equations are first made non-dimensional by appropriate combinations of a characteristic velocity U_*, a characteristic length scale L_*, and a characteristic temperature T_*. The choice of the characteristic scales, of course, depends upon the particular class of velocity and temperature profiles being considered. For symmetric flow in a channel bounded by the planes $y = \pm\frac{1}{2}d$ we non-dimensionalize the governing equations with respect to the maximum velocity of the basic flow at the centre of the channel, half the depth of the channel, and half the imposed temperature difference across the bounding planes. The equations governing the stability of the asymptotic suction boundary-layer profile are non-dimensionalized with respect to the velocity of the free stream, the displacement thickness δ_*, and the temperature difference between the free stream and the rigid boundary. For more general boundary-layer profiles it will be assumed that the velocity increases monotonically to a free stream value U_* at $y = \delta'_*$ and that for $y > \delta'_*$ the velocity and temperature are constant in the free stream. The boundary-layer thickness δ'_* then replaces the displacement thickness as the characteristic length.

The dimensionless numbers relevant to the stability of stably stratified viscous parallel flows are: the Reynolds number, the bulk Richardson number, and the Prandtl number, defined by

$$R \equiv \frac{U_* L_*}{\nu}, \quad Ri_b \equiv \frac{g\gamma T_* L_*}{U_*^2}, \quad \text{and} \quad P \equiv \frac{\nu}{k}, \tag{2.1}$$

respectively, where γ is the coefficient of thermal expansion. Although the bulk Richardson number emerges directly from the non-dimensionalization of the equations of motion, it does not measure the local behaviour of the velocity and temperature profiles. In the subsequent analysis, therefore, it will be necessary to introduce the local Richardson number,

$$Ri_l \equiv \frac{\Theta'}{(U')^2} Ri_b, \tag{2.2}$$

where U and Θ are the dimensionless basic velocity and temperature profiles. In this connexion, a parameter η will be introduced to represent the local Richardson number evaluated at the critical layer where the wave-speed equals the velocity of the undisturbed flow.

2.1. *The governing equations*

The stability of the basic flow to infinitesimal disturbances is governed by the Navier–Stokes, heat-conduction, and continuity equations. In addition, the usual approximate equation of state is required to relate density and temperature variations. To the lowest order these equations determine consistent profiles of velocity and temperature which for strictly parallel flow are assumed to depend only on y. To the next order these equations become the governing equations for the linearized stability theory. As usual, the disturbances will be assumed to be periodic in x and t and of the form $\exp\{i\alpha(x-ct)\}$, where α is the wave-number in the x-direction and c is the complex wave-speed. Throughout this paper the imaginary part of c has been set equal to zero in order to investigate the case of neutral stability.

Since it has been demonstrated by Gage & Reid (1968) that Squire's theorem is valid for the stability of stably stratified flows, it is sufficient to consider the two-dimensional problem of the stability of the basic flow to infinitesimal two-dimensional disturbances. A further simplification of the analysis is obtained by applying the Boussinesq approximation, i.e. neglecting variations in density except where they are multiplied by the gravitational acceleration.

The two-dimensional problem then consists of the equations

$$\left.\begin{aligned} \{D^2-\alpha^2-i\alpha R(U-c)\}u &= i\alpha Rp + RU'v, \\ \{D^2-\alpha^2-i\alpha R(U-c)\}v &= RDp - Ri_b R\theta, \\ i\alpha u + Dv &= 0, \\ \text{and} \qquad \{D^2-\alpha^2-i\alpha RP(U-c)\}\theta &= RP\Theta' v, \end{aligned}\right\} \tag{2.3}$$

with the boundary conditions,

$$u = v = \theta = 0, \tag{2.4}$$

to be imposed either at a rigid boundary or at infinity.

By introducing the stream function $\phi(y)\exp\{i\alpha(x-ct)\}$, with $u = \phi'$ and $v = -i\alpha\phi$, we can combine the governing equations into the form

$$L_4\phi = Ri_b\theta \quad \text{and} \quad L_2\theta = -\Theta'\phi, \tag{2.5}$$

where
$$\left.\begin{aligned} L_2 &\equiv (i\alpha RP)^{-1}(D^2-\alpha^2)-(U-c) \\ L_4 &\equiv (i\alpha R)^{-1}(D^2-\alpha^2)^2-(U-c)(D^2-\alpha^2)+U''. \end{aligned}\right\} \tag{2.6}$$
and

Eliminating θ in equations (2.5) we have the sixth-order ordinary differential equation

$$L_2L_4\phi + Ri_b\Theta'\phi = 0 \tag{2.7}$$

with the boundary conditions

$$\phi = \phi' = L_4\phi = 0 \quad \text{at} \quad y = \pm 1 \quad \text{or} \quad y = 0, \infty. \tag{2.8}$$

An equation of the type (2.7) has been studied by Hughes & Reid (1968) in connexion with the stability of spiral flow between rotating cylinders. The solutions presented by them include inviscid solutions of a second-order differential equation and viscous solutions which are asymptotic ($R \to \infty$, η fixed) approximations to the solutions of the full sixth-order differential equation. These latter solutions were developed first within the framework of the W.K.B. and local turning-point approximations, and subsequently combined to provide composite approximations of the Tollmien type. The W.K.B. method provides an outer approximation to the viscous solutions for the limit $\alpha R \to \infty$ with the critical point bounded away from the lower boundary. Inner approximations to the viscous solutions are provided within the framework of the local turning-point approximation. These are employed when the critical layer approaches the lower boundary as $\alpha R \to \infty$. Since small values of c are associated with a critical layer near the lower boundary, it follows that, with c small enough, it is sufficient to consider these inner approximations to the viscous solutions. When c is large enough to preclude the sufficiency of the inner viscous approximations, it is useful to employ approximations to the viscous solutions which reduce in the limit $\alpha R \to \infty$ to the local turning-point solutions or to the W.K.B. solution, depending upon whether the critical layer does or does not approach the lower boundary. These approximations represent composite solutions and are named after Tollmien, who introduced them into the stability theory for unstratified parallel flow.†

In the present study use was made of the asymptotic analysis presented in Hughes & Reid (1968). A brief outline of this analysis as it applies to the present work is presented in the following pages.

In order to develop approximations to the asymptotic solutions of (2.7) we first consider a formal expansion of the solution in inverse powers of $i\alpha R$:

$$\phi(y) = \phi^{(0)}(y) + (i\alpha R)^{-1}\phi^{(1)}(y) + \ldots, \tag{2.9}$$

where $\phi^{(0)}(y)$ satisfies the inviscid equation,

$$(U-c)^2(D^2-\alpha^2)\phi - (U-c)U''\phi + Ri_b\Theta'\phi = 0. \tag{2.10}$$

† See Reid (1965) for a detailed discussion of the behaviour of these solutions to the Orr–Sommerfeld equation.

The solutions of (2.10) can be represented in the form

$$\phi_1(y) = (y-y_c)^{p_1}P_1(y-y_c) \quad \text{and} \quad \phi_2(y) = (y-y_c)^{p_2}P_2(y-y_c), \tag{2.11}$$

where p_1 and p_2 are roots of the indicial equation

$$p(p-1)+\eta = 0, \tag{2.12}$$

with $p_1 > p_2$. $P_1(y-y_c)$ and $P_2(y-y_c)$ are power series in $(y-y_c)$ with leading terms of unity and with coefficients determined from the recursion relations developed in appendix 1 of Gage (1968).

The inviscid solutions (2.11) possess algebraic branch points at the critical layer y_c, and therefore cannot provide valid approximations in the full complex neighbourhood of y_c. Their range of validity can be deduced by studying the behaviour of the viscous solutions developed below. They provide approximations to the two viscous solutions which are neutral in the sector containing the boundary point. They can be valid only in those sectors of the complex plane where the complete expansion of the viscous solution is not dominant. This sector is $-7\pi/6 < \arg(y-y_c) < \pi/6$, which, of course, includes any boundary points.

2.2. *The viscous solutions*

The viscous approximations to the solutions of (2.7) will be developed below in the local turning-point approximation for $P = 1$. If c is large it is necessary to consider composite solutions of the Tollmien type. For this reason the characteristic equations employing both approximations are developed below. The local turning-point approximations, however, have the advantage of simplicity.

Consider, then, the transformation

$$\phi(y) = \chi(\xi), \quad \text{where} \quad \xi = (y-y_c)/\epsilon \quad \text{and} \quad \epsilon = (i\alpha R U_c')^{-\frac{1}{3}}, \tag{2.13}$$

and the expansion of the solution in powers of ϵ:

$$\chi(\xi,\epsilon) = \chi^{(0)}(\xi)+\epsilon\chi^{(1)}(\xi)+\ldots. \tag{2.14}$$

The first approximation $\chi^{(0)}(\xi)$ satisfies the equation

$$\{(D^2-\xi)^2D^2+\eta\}\chi = 0, \tag{2.15}$$

where D now represents $d/d\xi$. Equation (2.15), derived for $P = 1$, can now be factored, and it is sufficient to consider solutions of the two third-order equations

$$\chi'''-\xi\chi'+p_i\chi = 0, \tag{2.16}$$

where $p_i (i = 1, 2)$ are roots of the indicial equation (2.12).

Since it is expected that viscous effects will be small in the centre of a channel or far from a boundary, we need only consider the solutions of (2.15),

$$\chi_3(\xi) = A_1(\xi,p_1) \quad \text{and} \quad \chi_5(\xi) = A_1(\xi,p_2), \tag{2.17}$$

which are subdominant† in the sector $|\arg\xi| < \frac{1}{3}\pi$. The solutions $A_1(\xi,p_i)$ are then uniquely determined to within a multiplicative constant.

† These are the solutions defined in Hughes & Reid (1968) to (2.16), which are exponentially small far from the lower boundary.

2.3. *The characteristic equation*

Provided that $\Theta'(y)$ is an even function of y for symmetric channel flows the governing equations will be symmetric in y and it will be sufficient to treat the lower half of the channel. Further, the symmetry of the basic flow enables us to consider the even and odd solutions separately. Since we expect the onset of instability to be associated with an even solution, the boundary conditions at $y = 1$ will be replaced by

$$\phi' = \phi''' = \phi^{(v)} = 0 \quad \text{at} \quad y = 0. \tag{2.18}$$

Turning to the general boundary-layer situation described at the beginning of § 2 it is convenient to suppose that the velocity and temperature are continuous functions of y and monotonically increasing for $0 \leqslant y \leqslant 1$ with $U(y) = 1$ and $\Theta' = 0$ for $y > 1$. The inviscid solution valid in the free stream which satisfies the boundedness condition at $y = \infty$ is then of the form

$$\Phi = K e^{-\alpha y}, \tag{2.19}$$

where the constant K is determined (below) by matching Φ and Φ' at $y = 1$.

If we let $y_1 = -1, y_2 = 0$ and $y_1 = 0, y_2 = 1$ for the channel and boundary-layer flows, respectively, it is possible to formulate the characteristic value problem in a unified manner. It is convenient for this purpose to let

$$\Phi = A\phi_1 + \phi_2 \tag{2.20}$$

be the inviscid solution of (2.9) in the interval $y_1 \leqslant y \leqslant y_2$.

For channel flows this solution is valid† at both the centre and the lower boundary. Since viscous effects are expected to be negligible at the centre, A is determined by satisfying the boundary condition $\Phi'(0) = 0$. The other two boundary conditions (2.18) are then satisfied with an exponentially small error.

Similarly, for the class of boundary-layer flows described above, A is determined by the matching conditions

$$K e^{-\alpha} = A\phi_1(1) + \phi_2(1) \tag{2.21}$$

and

$$-\alpha K e^{-\alpha} = A\phi_1'(1) + \phi_2'(1), \tag{2.22}$$

so that

$$K = \frac{e^{\alpha}}{\alpha\phi_1(1) + \phi_1'(1)} \quad \text{and} \quad A = -\frac{\alpha\phi_2(1) + \phi_2'(1)}{\alpha\phi_1(1) + \phi_1'(1)}. \tag{2.23}$$

The characteristic equation is obtained from the simultaneous satisfaction of the boundary conditions (2.8) at y_1. If we take an approximation to the general solution of the form

$$\phi = \Phi + C_3\chi_3 + C_5\chi_5, \tag{2.24}$$

then the first two conditions are

$$\Phi(y_1) + C_3\chi_3(\xi_1) + C_5\chi_5(\xi_1) = 0 \tag{2.25}$$

and

$$\Phi'(y_1) + C_3\epsilon^{-1}\chi_3'(\xi_1) + C_5\epsilon^{-1}\chi_5'(\xi_1) = 0. \tag{2.26}$$

† Both ϕ_1 and ϕ_2 are valid approximations to two of the viscous solutions.

The third condition requires further consideration and will be given below in an approximate form. Consider first the formal application of the operator L_4 to the inviscid function Φ. With the aid of (2.10) this operation yields

$$L_4\Phi = \epsilon^3 U_c'(D^2-\alpha^2)^2\Phi + \frac{Ri_b\,\Theta'}{U-c}\Phi.$$

Thus, consistent with the approximations already made,

$$L_4\Phi \to \frac{\Theta' Ri_b}{U-c}\Phi. \tag{2.27}$$

Formal application of the operator L_4 to the viscous function χ leads to

$$\{U_c'\epsilon^{-1}(\chi^{\text{iv}}-\xi\chi'')+O(1)\}\chi = 0,$$

or

$$\{U_c'\epsilon^{-1}(1-p)\chi'+O(1)\}\chi = 0,$$

upon reference to (2.16). Again consistent with the present approach,

$$L_4\chi \to \epsilon^{-1}U_c'(1-p)\chi' \tag{2.28}$$

and finally setting $L_4\phi$ equal to zero implies

$$-\frac{\Theta'(y_1)}{c}Ri_b\Phi(y_1)+C_3\epsilon^{-1}U_c'(1-p_1)\chi_3'(\xi_1)+C_5\epsilon^{-1}U_c'(1-p_2)\chi_5'(\xi_1) = 0. \tag{2.29}$$

A non-trivial solution of the homogeneous system, (2.25), (2.26), and (2.28), requires that the determinant of the C_i vanishes,

$$\begin{vmatrix} \Phi(y_1) & \chi_3(\xi_1) & \chi_5(\xi_1) \\ \Phi'(y_1) & \epsilon^{-1}\chi_3'(\xi_1) & \epsilon^{-1}\chi_5'(\xi_1) \\ -\dfrac{U_c'\eta\Theta'(y_1)}{c\Theta'(y_c)}\Phi(y_1) & \epsilon^{-1}(1-p_1)\chi_3'(\xi_1) & \epsilon^{-1}(1-p_2)\chi_5'(\xi_1) \end{vmatrix} = 0. \tag{2.30}$$

Expansion and simplification then leads to the characteristic equation in the form

$$\Delta(\alpha,c,z;\eta) \equiv \frac{1}{y_c-y_1}(p_1-p_2)+\frac{\Phi'(y_1)}{\Phi(y_1)}\{p_1F(z,p_1)-p_2F(z,p_2)\}$$
$$+\frac{\theta'(y_1)U_c'}{\theta'(y_c)c}p_1p_2\{F(z,p_1)-F(z,p_2)\} = 0, \tag{2.31}$$

where $z = (\alpha R U_c')^{\frac{1}{3}}(-y_1+y_c)$ and $F(z,p)$ is the generalized Tietjens function

$$F(z,p) \equiv \frac{A_1(\xi,p)}{\xi A_1'(\xi,p)} \tag{2.32}$$

with $\xi = ze^{-5\pi i/6}$ and p real.

If the critical layer is sufficiently removed from the lower boundary, better approximations to the viscous solutions are obtained by using composite solutions of the Tollmien type (see Hughes & Reid 1968, ch. 10). When these composite solutions are introduced into the characteristic equation, we obtain

$$\Delta'(\alpha,c,z;\eta) \equiv \frac{4\sqrt{c}}{3\mu(c)}(p_1-p_2)+\frac{\Phi'(y_1)}{\Phi(y_1)}\{p_1F(\hat{z},p_1)-p_2F(\hat{z},p_2)\}$$
$$+\frac{3\Theta'(y_1)}{4\Theta'(y_c)}\mu(c)c^{-\frac{5}{2}}(U_c')^2p_1p_2\{F(\hat{z},p_1)-F(\hat{z},p_2)\} = 0, \tag{2.33}$$

where $$\hat{z} = (\alpha R)^{\frac{1}{3}} \left\{\frac{3\mu(c)}{4}\right\}^{\frac{2}{3}} \quad \text{and} \quad \mu(c) = 2\int_{y_1}^{y_c} |U-c|^{\frac{1}{2}} dy, \tag{2.34}$$

which reduces to (2.30) for small values of c. As we shall see in §3, the universal viscous limit with $c \to 0$ emerges in a natural way from (2.31).

3. Results for flows without inflexion points

The analysis reviewed in §2 was employed in treating the stability of two stratified symmetric channel flows and one stratified boundary-layer flow. The results for plane Poiseuille flow and the asymptotic suction boundary-layer profile are presented below. Results for Grohne's inflexion point profile will be considered in §4. Although no examples of the general boundary-layer situation defined in §2 were treated in detail, the class of profiles defined there satisfy the universal viscous limit discussed below.

3.1. *Plane Poiseuille flow*

The results for stratified plane Poiseuille flow with velocity and temperature profiles
$$U(y) = 1-y^2 \quad \text{and} \quad \Theta(y) = y \tag{3.1}$$
were presented by Gage & Reid (1968). The curves of neutral stability obtained from the numerical solution of the characteristic equation were shown there in figure 2 and the corresponding neutral curves for the wave-speed c were shown in figure 3. The neutral curve for $\eta = 0$ shows the known results without stratification, and the appearance of the kink in the upper branch is associated with the loop in the Tietjens function.

The effect of stable stratification is to increase the minimum critical Reynolds number and to decrease the region of instability bounded by the neutral curve. It was found that $c \to 0$, $\alpha \to \alpha_s(\eta)$, and $z \to z_s^{\pm}(\eta)$ as $R \to \infty$ on both branches of the neutral curves. Finally, by the results of table 1 of Gage & Reid (1968), also plotted there in figure 4, the critical Reynolds number approaches infinity as $\eta \to 0{\cdot}0554$.

3.2. *The asymptotic suction boundary layer*

The stability of the unstratified asymptotic suction boundary-layer profile has been studied within the framework of the asymptotic theory by Hughes & Reid (1965). This flow is of special significance because it is possible to obtain the solutions of the inviscid equation analytically in terms of hypergeometric functions.

For $P = 1$ the basic velocity and temperature profiles that satisfy the momentum and heat conduction equations are given by (Miles 1967)

$$U(y) = 1-e^{-y}, \quad V = -1/R \quad \text{and} \quad \Theta(y) = \frac{1-e^{-\lambda}}{1-e^{-\sigma}}, \tag{3.2}$$

where $$\lambda \equiv \log[\rho_0/\rho] = \sigma(1-e^{-y}) \quad \text{and} \quad |\sigma| \ll 1. \tag{3.3}$$

In these equations V is the suction velocity and $R = U_* \delta_* / \nu$ in keeping with the use of the displacement thickness as characteristic length. The asymptotic suction boundary-layer profile is an almost parallel flow with the suction velocity slightly modifying the governing equations. Since it can be shown that these modifications occur at an order higher than the order of terms retained in the present asymptotic analysis, it is possible to treat the asymptotic suction boundary layer consistently as a parallel flow.

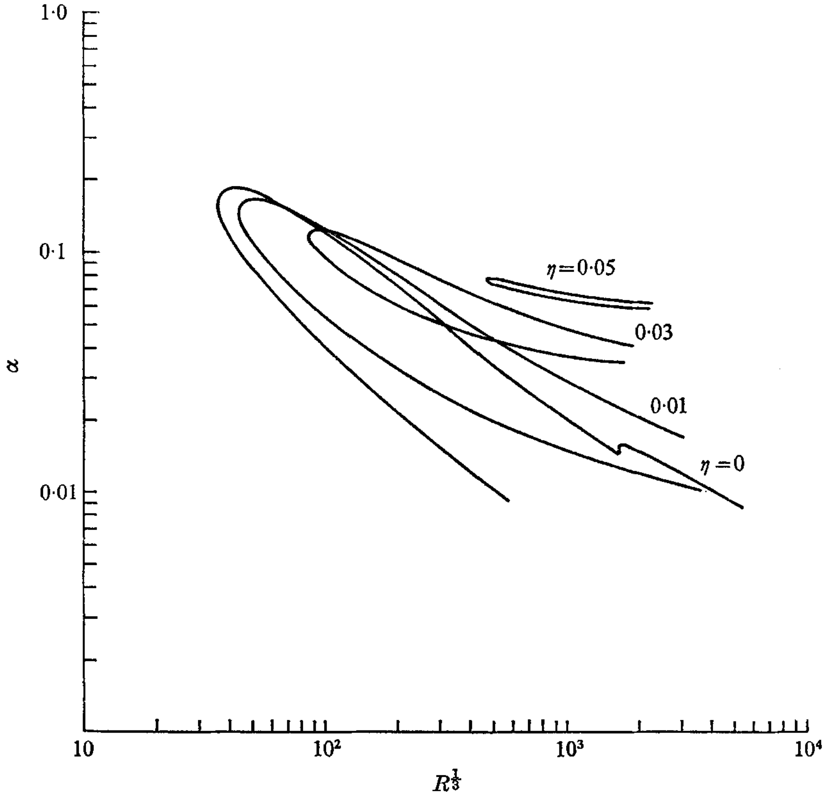

FIGURE 1. The curves of neutral stability for the stably stratified asymptotic suction boundary-layer profile.

The exact solutions of the inviscid equation (2.9) for the stratified asymptotic suction boundary-layer profile with $\Theta' = e^{-y}$ are developed in the appendix.

Figures 1 and 2 show the curves of neutral stability obtained for the asymptotic suction boundary-layer profile at several values of η. Comparison with the corresponding figures of Gage & Reid (1968) demonstrates the qualitative similarity of the effect of stable stratification upon the boundary layer, and upon the channel flow considered here. In fact, with $\eta > 0$ our calculations confirm that the viscous limits with $c \to 0$, $\alpha \to \alpha_s$ and $z \to z_s^{\pm}$ as $R \to \infty$ along both branches of the neutral curves are of the same form for both flows. Finally, from the results in table 1 and figure 3 we see that the minimum critical Reynolds number again approaches infinity as $\eta \to 0{\cdot}0554$.

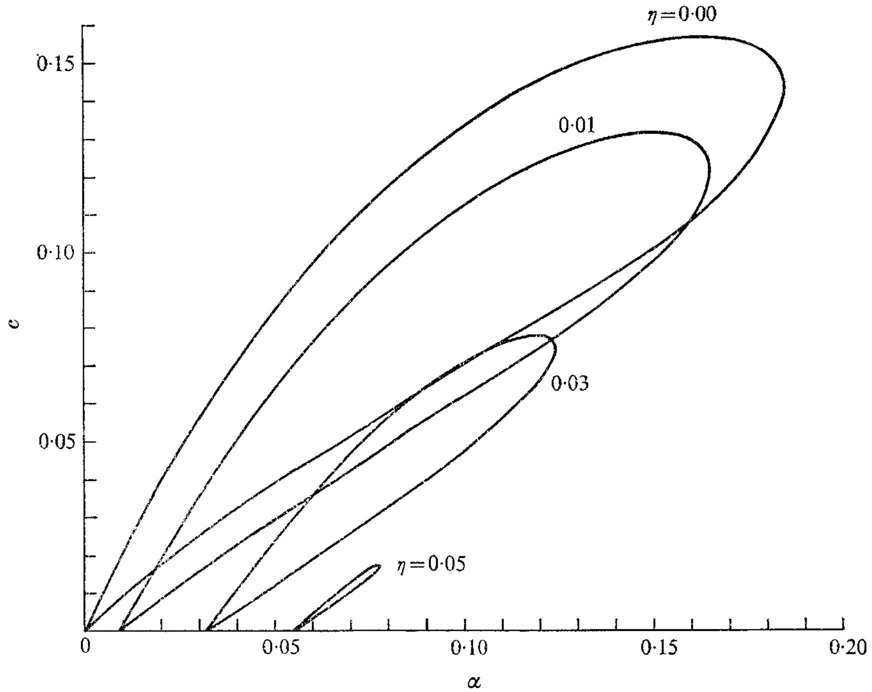

FIGURE 2. The relationship between the wave-number α and the wave-speed c along the neutral curves of figure 4.

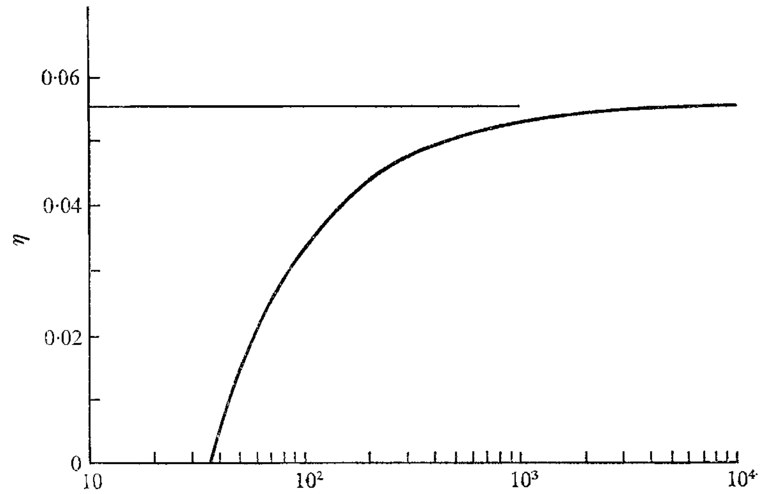

FIGURE 3. The variation of the minimum critical Reynolds number with η for the asymptotic suction boundary-layer profile.

η	Ri_b	z	α	c	$R_{cr}^{\frac{1}{3}} \times 10^{-2}$
0·0100	0·0087	3·18	0·147	0·131	0·4479
0·0200	0·0179	3·24	0·1335	0·1052	0·5913
0·0300	0·0277	3·30	0·1176	0·0776	0·8559
0·0350	0·0328	3·33	0·1087	0·0632	1·091
0·0390	0·0370	3·36	0·1014	0·0514	1·388
0·0430	0·0413	3·38	0·0930	0·0393	1·884
0·0470	0·0457	3·41	0·0844	0·0270	2·870
0·0500	0·0491	3·44	0·0771	0·0175	4·596
0·0510	0·0503	3·45	0·0746	0·0143	5·704
0·0520	0·0514	3·46	0·0719	0·0111	7·473
0·0528	0·0523	3·46	0·0698	0·0087	9·687
0·0554	0·0554	3·48	—	0·0000	∞

TABLE 1. The values of the minimum critical Reynolds number and related parameters for the stably stratified asymptotic suction boundary-layer profile

3.3. *The universal viscous limit of the characteristic equation*

In the theory for the stability of homogeneous parallel shear flow two different kinds of limits are achieved as viscosity vanishes along the curves of neutral stability. The first class consists of limits where $\alpha R \to \infty$, such that the critical layer does not approach the rigid boundary. These limits are correctly referred to as inviscid limits and occur only in flows with an inflexion point in the basic velocity profile. The second class consists of limits where $\alpha R \to \infty$, such that the critical layer approaches the lower boundary. This limit is referred to here as a *viscous limit*, provided the viscous terms in the characteristic equation (2.31) are of order unity in the limit $y_c \to y_1$ or, equivalently, $c \to 0$. Viscous terms are retained therefore in the limit $\epsilon \to 0$ whenever z (or $\xi_1 = (y_1 - y_c)/\epsilon$) remains finite. A limit of this type is expected to be associated with the weak viscous Tollmien–Schlichting mechanism.

For homogeneous parallel flows the viscous limit which occurs along the lower branch of the curve of neutral stability is known to lead to a universal result, $z \to 2{\cdot}294$ as $c \to 0$, independent of the basic velocity profile. It is demonstrated below for stratified flow that this universal behaviour occurs in viscous limits along *both* branches of the curves of neutral stability for $0 < \eta < 0{\cdot}0554$. Universality is guaranteed, since only the leading terms in the expansion of the inviscid solutions about the critical point enter the limiting form of the characteristic equation.

As the critical point approaches the lower boundary for fixed η, the characteristic equation (2.31) obtained for symmetric channel and monotonic boundary-layer flows has the limiting form

$$p_1 - p_2 + Q\{p_1 F(z, p_1) - p_2 F(z, p_2)\} + p_1 p_2\{F(z, p_1) - F(z, p_2)\} = 0, \quad (3.4)$$

where

$$Q = \lim_{y_c \to y_1} \left\{\frac{(y_c - y_1)\Phi'(y_1)}{\Phi(y_1)}\right\}. \quad (3.5)$$

In (3.4) only Q can depend on the specific velocity and temperature profiles.

However, as $c \to 0$ and $y_c \to y_1$, it follows from (2.11) that

$$\phi_1(y_1) \to \exp(-i\pi p_1)(y_c - y_1)^{p_1} \quad \text{and} \quad \phi_2(y_1) \to \exp(-i\pi p_2)(y_c - y_1)^{p_2}, \tag{3.6}$$

so that Q has the limiting form

$$Q = -\frac{Ap_1 \exp(-i\pi p_1)(y_c - y_1)^{p_1} + p_2 \exp(-i\pi p_2)(y_c - y_1)^{p_2}}{A \exp(-i\pi p_1)(y_c - y_1)^{p_1} + \exp(-i\pi p_2)(y_c - y_1)^{p_2}}. \tag{3.7}$$

Furthermore, consistent with a viscous limit, for finite values of z, as $R \to \infty$ and $c \to 0$ on both branches of the neutral curves, we have

$$A \sim a_s^{\pm}(y_c - y_1)^{-(p_1 - p_2)}, \tag{3.8}$$

where $a_s^{\pm}$ are eigenvalues (associated with $z_s^{\pm}$) of the characteristic equation in the viscous limit. Thus, in the limit $y_c \to y_1$, Q can be expressed in the form

$$Q(a_s^{\pm}, \eta) = -\frac{a_s^{\pm} p_1 \exp(-i\pi p_1) + p_2 \exp(-i\pi p_2)}{a_s^{\pm} \exp(-i\pi p_1) + \exp(-i\pi p_2)}. \tag{3.9}$$

In order to obtain the limiting values of z along both branches of the curve of neutral stability for any given value of η it is necessary to solve the limiting form (3.4) of the characteristic equation. Table 2 contains the results of this calculation, and figure 4 shows the variation of $z_s^{\pm}$ with η.

η	a_s^-	z_s^-	a_s^+	z_s^+
0·0000	2·296	2·297	1·000	∞
0·0100	2·186	2·420	1·005	5·073
0·0200	2·068	2·555	0·977	4·645
0·0300	1·938	2·707	0·983	4·378
0·0400	1·781	2·887	1·017	4·136
0·0500	1·572	3·134	1·109	3·850
0·0520	1·512	3·207	1·143	3·772
0·0540	1·433	3·307	1·196	3·667
0·0554	1·31	3·48	1·31	3·48

TABLE 2. The values of the parameters associated with the universal viscous limit along the curves of neutral stability

Since the wave-number does not appear in the limiting form of the characteristic equation, it is necessary to consider how it is determined. With $A \to \infty$ according to (3.8) all conditions at the rigid boundary are satisfied in the viscous limit of the characteristic equation.† Requiring the inviscid solution ϕ_1 to satisfy the condition $\phi_1'(y_2) = 0$ determines the limiting value of the wave-number. Since the inviscid solutions depend upon the basic velocity and temperature profiles, these limiting values are not universal.

Returning to the universal viscous behaviour of the eigensolutions of (3.4) shown in figure 7 we observed that as $\eta \to 0{\cdot}0554$ the limiting values of z on the upper and lower branches approach a common limit. This fact implies the

† $A \to \infty$ is consistent with a theorem of Miles (1961) from the inviscid theory of stratified flow. If the Reynolds stress is zero, A must be zero or infinity. The viscous nature of the limit $R \to \infty$ at the lower boundary precludes application of Miles's result there.

minimum critical Reynolds number approaches infinity as this common limit is achieved. Physically, this means that a weak instability, purely viscous in origin, is completely stabilized as $\eta \to 0{\cdot}0554$ from below.

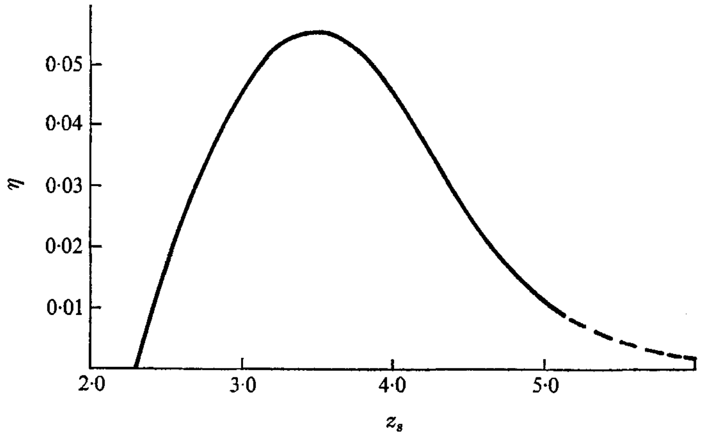

FIGURE 4. The variation with η of the limiting values of z on both branches of the neutral curves.

4. Results for a flow with an inflexion point

In § 4 we consider the stability of Grohne's (1954) inflexion-point profile with thermal stratification. This profile was chosen because of its simple analytical form. It is a symmetric channel flow with dimensionless basic velocity

$$U(y) = (\sqrt{2}-1)+(2-\sqrt{2})\cos 3\pi y/4. \tag{4.1}$$

For convenience we have taken the temperature profile to be linear, i.e.

$$\Theta(y) = y. \tag{4.2}$$

As pointed out in § 1, considerable work has been done investigating the stability of stably stratified parallel flow of an inviscid fluid. Essentially, this work involves the solution of a characteristic value problem† of the form

$$F(\alpha, c; \eta) = 0.$$

Figures 5 and 6 show the result of such a calculation for Grohne's profile. On the basis of these inviscid results, complete stabilization is predicted for $\eta = 0{\cdot}0773$.

The results for the viscous analysis using Tollmien composite solutions and the characteristic equation (2.32) for the stability of the inflexion point profile are presented in figures 8–10. The special values of η which emerge from the viscous limit of § 3 and the inviscid calculations described above conveniently divide η into three distinct ranges.

In the first range, $0 < \eta < 0{\cdot}0554$, there is a weak instability, viscous in origin, coexistent with a strong instability associated with the inflexion point. The

† For the purposes of the present paper we required $\Phi'(y_2) = \Phi(y_1) = 0$. No attempt was made to show the resulting neutral curves to be stability boundaries.

effect of the strong instability is that the neutral curves approach inviscid limits with $z \to \infty$ as $R \to \infty$, thereby recovering the inviscid eigenvalues α_s, c_s along two branches. In this limit the inviscid eigensolution Φ_s is a multiple of ϕ_2. However, the weak, purely viscous instability recovers the universal viscous limit discussed in § 3, with $z \to z_s$, as $R \to \infty$ along another two branches of the neutral curves. These considerations require the existence of a *minor curve* of neutral stability.

Figure 7 (*a*) contains a schematic diagram of the major and minor curves of neutral stability for a fixed value of η in the first range. The lower branch of the

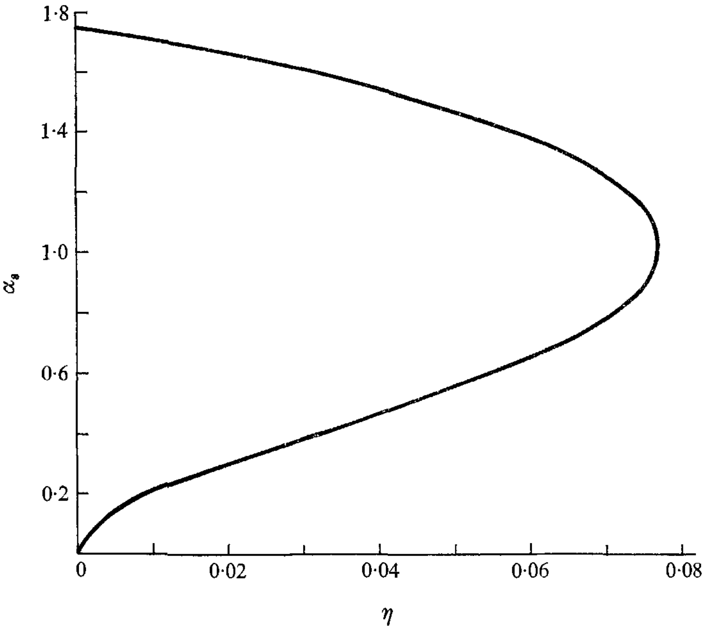

FIGURE 5. The dependence of the inviscid eigenvalue for the wave-number α upon η for Grohne's profile.

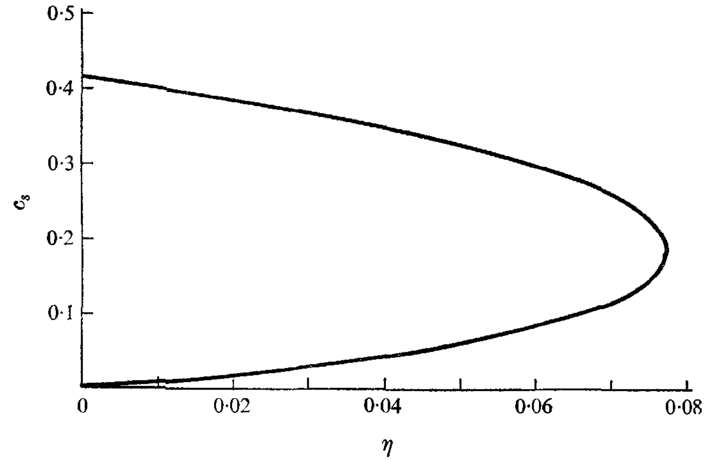

FIGURE 6. The dependence of the inviscid eigenvalue for the wave-speed c upon η for Grohne's profile.

major curve and the lower branch of the minor curve correspond respectively to the lower and upper branches of the neutral curves for profiles without inflexion points. Furthermore, the limiting values of z on both of these branches must come from the solution of the characteristic value problem in the universal viscous limit $c \to 0$ discussed in § 3. The upper branches of the major and minor curves recover the inviscid eigensolutions of figures 5 and 6 in an inviscid limit. Finally, as $\eta \to 0{\cdot}0554$, the lower branches of the major and minor curves join together.

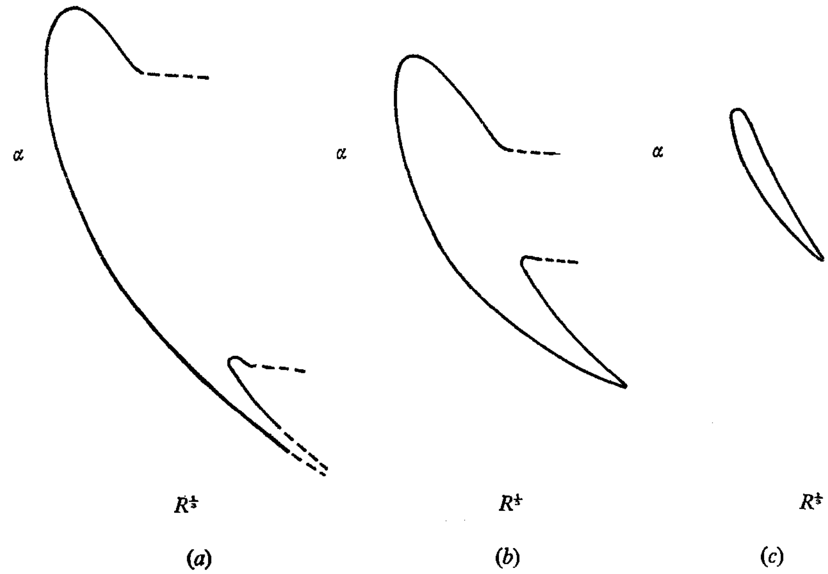

FIGURE 7. Schematic diagrams of the neutral curves for Grohne's inflexion-point profile in the three ranges of η.

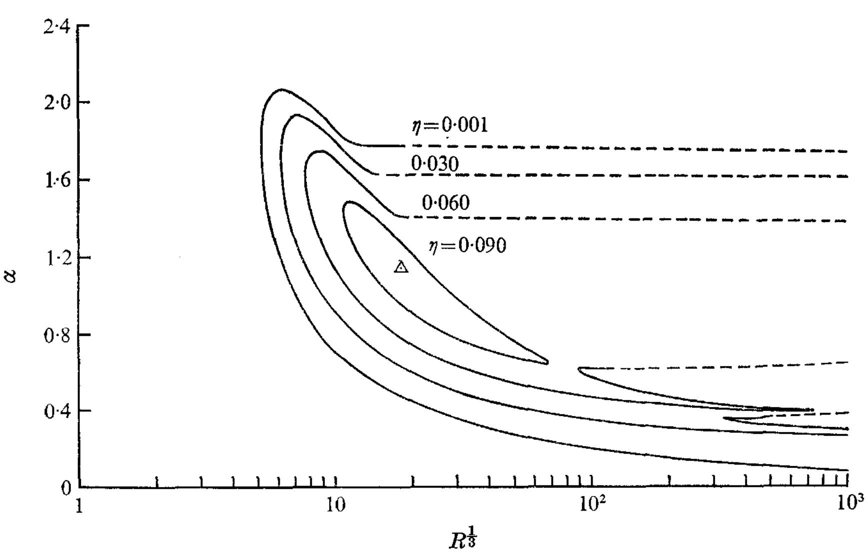

FIGURE 8. The curves of neutral stability for Grohne's profile with stable stratification. △, complete stabilization.

For $0{\cdot}0554 < \eta < 0{\cdot}0773$ we have a continuous neutral curve (see schematic diagram in figure 7 (*b*)) the upper branch of which corresponds to the upper branch of the old major curve and the lower branch of which corresponds to the upper branch of the old minor curve. On both the upper and lower branches, inviscid limits are attained as $z \to \infty$ and the inviscid eigensolutions are recovered.

As $\eta \to 0{\cdot}0773$ the neutral curves close up, but there is a residual instability (see schematic diagram in figure 7 (*c*)). As η increases beyond that value, the region of instability contained within the closed neutral curves decreases until complete stabilization is achieved at a finite Reynolds number, as indicated in figure 10.

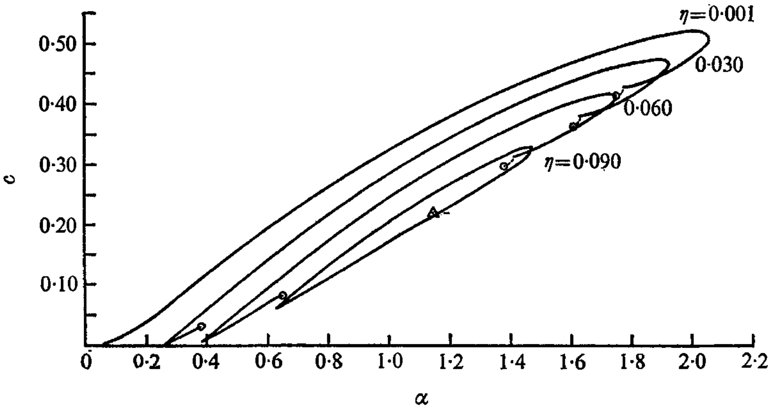

FIGURE 9. The relationship between the wave-number α and the wave-speed c along the neutral curves of figure 7. △, complete stabilization.

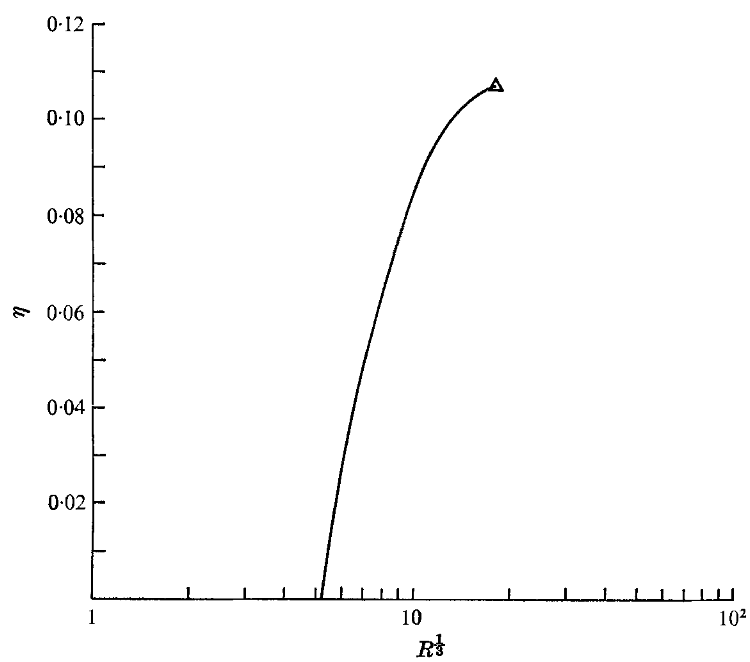

FIGURE 10. The variation with η of the minimum critical Reynolds number for Grohne's profile. △, complete stabilization.

5. Concluding remarks

In § 3 and § 4 we have obtained some fairly general results for the stability of stably stratified, viscous, parallel shear flows in the presence of at least one rigid boundary. When the velocity profile had no inflexion point, we found that there exists a universal value of η for which a weak instability of viscous origin is completely stabilized. However, for profiles with an inflexion point, the results for Grohne's profile demonstrate that the limiting value of η for complete stabilization of the viscous flow may be somewhat greater than that required to stabilize the corresponding inviscid flow.

All of these results have been developed within the framework of the asymptotic ($R \to \infty$) theory for the stability of parallel shear flows to infinitesimal disturbances. The Boussinesq approximation has been made and only the case of $P = 1$ has been treated.

The fact that our limiting value of $\eta = 0{\cdot}0554$ for complete stabilization of flows without an inflexion point is close to the value 0·0409, which was found by Schlichting (1935) for the Blasius profile, is somewhat surprising in view of the differences between the two approaches. As pointed out in § 1, Schlichting ignored heat conduction, so that he treated a fourth-order governing differential equation as appropriate for a fluid of infinite Prandtl number. Although it would be tempting to conclude from Schlichting's result that the limiting value of η varies weakly with the Prandtl number, upon closer examination it appears that the assumption of negligible heating due to viscous dissipation, made in both approaches, would be inconsistent for a fluid of infinite Prandtl number.

Very little experimental work has been done on the stability of thermally stratified parallel shear flows. Perhaps the best experimental results available are those of Reichardt, reported by Schlichting (1935). Reichardt investigated the transition to turbulence in developing boundary layers on the upper and lower plates of a wind-tunnel 16 m long with a rectangular cross-section 1 m wide and 25 cm deep. The flow was stably stratified by heating the upper plate with steam and cooling the lower plate with water (10 °C). Because the temperature of the incoming air was closer to the temperature of the lower plate, the profiles of temperature and velocity were asymmetric, with a more stable stratification under the upper plate. These velocity and temperature profiles were measured by hot-wire probes. The decision as to whether the flow was laminar or turbulent was made with reference to the amplitude of oscillations measured by these probes and recorded on an oscillograph.

In order to make any meaningful comparison between theory and experiment, it would be necessary to plot experimental points of R and Ri_b, say, on the graph for the corresponding theoretical stability boundary. Since the boundary layers in Reichardt's work are essentially Blasius boundary layers, which were not treated here, one can do no more than point out that Reichardt's results are consistent with the complete stabilization of an uninflected velocity profile when the local Richardson number exceeds 0·0554 throughout the flow.

Schlichting did treat the stratified Blasius boundary-layer profile and he did make a comparison between his theoretical results and the observations of

Reichardt. This was accomplished by fitting theoretical velocity and temperature profiles to the observed profiles, computing the local Richardson number at the boundary and the Reynolds number of the flow, and plotting each point on the graph containing the theoretical stability boundary, noting whether the flow was laminar or turbulent. Although the result of these calculations (shown in figure 15 of Schlichting 1935) were consistent with Schlichting's theoretical predictions, not enough experimental points were included for a definitive verification of the theoretical stability boundary. In this connexion there appears to have been some experimental difficulty in obtaining Reynolds numbers much above 2000, and, for that reason, instability was not observed when the local Richardson number evaluated at the boundary was more than 0·022.

The next logical step in developing the theory for the stability of stratified viscous parallel flow is to consider unbounded flow of the jet or shear-layer type. The stability characteristics are expected to differ somewhat from the bounded and semi-bounded flows treated here. Specifically inviscid eigensolutions should be recovered on both branches of a continuous neutral curve for any $\eta > 0$ but less than the value of η which stabilizes the inviscid flow. It is conjectured therefore that the viscous limits reported here are relevant only to bounded or semi-bounded flows.

The investigation of the stability of a stratified jet may be of relevance to the study of a generating mechanism for clear-air turbulence associated with the jet stream. In this connexion, Ludlam (1967) has reported the appearance of billow clouds in a region just below the axis of a jet stream where the averaged local Richardson number was determined to be $\frac{1}{4}$. At the present time, most observations concerning clear-air turbulent phenomena are being made in regions of developed turbulence. Before any critical evaluation can be made of the relevance of shear flow instability to the generation of clear-air turbulence, observations of velocity and temperature profiles will have to be made upstream in regions of initial instability.

In conclusion, theoretical results have been presented in this paper for a large class of stably stratified viscous parallel shear flows. Although the available experimental results are consistent with the theory, no critical test has been provided. I can only hope that the results presented here will in the near future stimulate further experimental work on this problem, and that these results will be of some use to meteorologists concerned with stability mechanisms associated with the atmospheric boundary layer and the jet stream.

The bulk of the research reported in this paper was presented as a Ph.D. dissertation in the Department of the Geophysical Sciences at the University of Chicago under the supervision of Professor W. H. Reid. Financial support was provided there by the National Science Foundation (GK-944) and by the U.S. Navy (N 00014-67-0285-0002). Computations were completed at the University of Maryland with the financial support of the National Aeronautics and Space Administration Grant (NSG-398) to the Computer Science Center.

Appendix. The exact solutions of the inviscid equation for the stratified asymptotic suction boundary layer

Consider the inviscid equation (2.10) with $\Theta' = e^{-y}$ and let

$$\phi(y) = e^{-\alpha y} f(t),$$

where
$$t = t_0 e^{-y} \quad \text{and} \quad t_0 = 1/1-c. \tag{A 1}$$

The inviscid equation then becomes

$$f'' + \frac{1+2\alpha}{t} f' + \frac{1}{(1-t)t}\left[1 + \frac{Ri_b t_0}{1-t}\right] f = 0, \tag{A 2}$$

where the primes now denote differentiation with respect to t. A further transformation is required to get equation (A 2) in the form of Gauss's equation, and for this purpose we take

$$f(t) = (1-t)^{\gamma} F(t), \tag{A 3}$$

so that $F(t)$ then satisfies the equation

$$t(1-t)F'' + \{(1+2\alpha)(1-t) - 2\gamma t\}F' + \{1-(1+2\alpha)\gamma + \eta\}F = 0, \tag{A 4}$$

where η is still the local Richardson number evaluated at the critical point and is equal to $Ri_b/(1-c)$ provided γ is identified with p from the power series solution.

Since the solution of (A 4) has to be evaluated at $t_0 = 1/(1-c) > 1$ and since the hypergeometric equation has a branch point at $t = 1$, it is necessary to obtain the analytic continuation of the solution of (A 4). The solution can be given in terms of Gauss's hypergeometric function; and, with the proper analytic continuation (see, e.g. Erdelyi *et al.* 1953), we have

$$F(p,q,r;t) = A_1 F(p,q,p+q-r+1;1-t) + A_2(1-t)^{(r-p-q)} F(r-p,r-q,r-p-q+1;1-t), \tag{A 5}$$

where
$$A_1 = \frac{\Gamma(r)\,\Gamma(r-p-q)}{\Gamma(r-p)\,\Gamma(r-q)} \quad \text{and} \quad A_2 = \frac{\Gamma(r)\,\Gamma(p+q-r)}{\Gamma(p)\,\Gamma(q)} \tag{A 6}$$

and
$$p \equiv \alpha + \gamma + \{1+\alpha^2\}^{\frac{1}{2}},$$
$$q \equiv \alpha + \gamma - \{1+\alpha^2\}^{\frac{1}{2}},$$
$$r \equiv 1 + 2\alpha.$$

Finally, the solution $f(t)$ and its first derivative evaluated at the boundary, $t = t_0$, are given by

$$f(t_0) = A_1(1-t_0)^{\gamma} F(p,q,p+q-r+1;1-t_0) + A_2(1-t_0)^{(1-\gamma)} F(r-p,r-q,r-p-q+1;1-t_0) \tag{A 7}$$

2-2

and

$$f'(t_0) = -A_1\gamma(1-t_0)^{(\gamma-1)}F(p,q,p+q-r+1;1-t_0)$$
$$-A_1\left\{\frac{pq}{p+q-r+1}\right\}(1-t_0)^{\gamma}F(p+1,q+1,p+q-r+2;1-t_0)$$
$$-A_2(1-\gamma)(1-t_0)^{-\gamma}F(r-p,r-q,r-p-q+1;1-t_0)$$
$$-A_2\left\{\frac{(r-p)(r-q)}{r-p-q+1}\right\}(1-t_0)^{(1-\gamma)}F(r-p+1,r-q+1,r-p-q+2;1-t_0), \tag{A8}$$

where the correct branch is specified by taking

$$1-t = (t-1)e^{\mp i\pi}(t \gtrless 1). \tag{A9}$$

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